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Lattice trees with specified topologies

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Abstract. In this paper we study the total numbers of lattice trees *with specified topologies*. For strongly embeddable (or site) clusters with n_i branching points of degree i, we show how to prove rigorously that the growth constants exist and are all equal to the neighbour-avoiding walk limit υ. (This extends earlier work by Lipson and Whittington who proved an analogous result for weakly embeddable (or bond) clusters, for which the corresponding growth constant is the self-avoiding walk limit μ.) We derive some exact upper bounds for the critical exponents associated with the 'star', 'comb' and 'brush' topologies.

Exact enumeration data are derived and analysed for both weak and strong embeddings of some stars, combs and brushes on the square, triangular, simple cubic and $d = 4$ simple hypercubic lattices. We argue that the universality class for lattice trees with specified topology depends on the number, b, of branches, *possibly* through the conjectured critical exponent $(\gamma + b - 1)$, but not on any other details of the topology. Here γ is the critical exponent associated with self-avoiding walks.

We have also derived some exact enumeration data for the general d-dimensional simple hypercubic lattice. Using these data and the exact results for the interior of a Bethe lattice, we derive expansions for the growth constants in inverse powers of the dimensionality. These results are consistent with the growth constants being equal to the appropriate walk limits (μ or υ).

We discuss the relationship of our work to renormalisation group results which suggest that the universality class of branched polymers is independent of the branching fugacity.

1. Introduction

Branched polymer molecules with excluded volume have been modelled as lattice animals, i.e. as connected clusters embeddable in a regular lattice (Lubensky and Isaacson 1979). A number of workers (Lubensky and Isaacson 1979, Family 1980, Daoud and Joanny 1981) have used renormalisation group ideas to discuss the importance of cycles on their properties, arguing that the universality class is independent of the cycle fugacity. To investigate this point further, lattice trees (i.e. connected clusters having no cycles) have been studied using series expansion methods (Duarte and Ruskin 1981, Gaunt *et al* 1982) and Monte Carlo techniques (Seitz and Klein 1981). This evidence suggests that lattice trees and lattice animals are in the same universality class. So, for example, suppose that the number, a_n, per lattice site of weakly embeddable lattice animals with n vertices has the usual asymptotic form

$$a_n \sim n^{-\theta}\lambda^n \qquad (n \to \infty) \tag{1.1}$$

where λ is the growth constant for animals and θ is the associated critical exponent, and let the analogous form for the number, a_{n0}, per lattice site of weakly embeddable lattice trees with n vertices be

$$a_{n0} \sim n^{-\theta_0} \lambda_0^n \qquad (n \to \infty). \tag{1.2}$$

Then one finds (Gaunt *et al* 1982), for simple hypercubical lattices of coordination number $q = 2d$ and dimensionality $d = 2, 3, \ldots, d_c$ where d_c ($= 8$) is the upper critical dimension, that

$$\theta_0 = \theta \tag{1.3}$$

and, incidentally, that

$$\lambda_0 < \lambda. \tag{1.4}$$

The result (1.3) appears to support the renormalisation group contention that the universality class is independent of the cycle fugacity.

In order to investigate the crossover from trees to animals, Whittington *et al* (1983) considered the numbers, a_{nc}, per lattice site of weakly embeddable clusters with n vertices and cyclomatic index c. Assume that asymptotically the number of such c-animals goes like

$$a_{nc} \sim n^{-\theta_c} \lambda_c^n \qquad (n \to \infty) \tag{1.5}$$

for all c. Whittington *et al* (1983) have shown rigorously that the growth constant λ_c is independent of c, i.e.

$$\lambda_c = \lambda_0 \qquad (c = 0, 1, 2, \ldots) \tag{1.6}$$

and have presented convincing evidence that the associated critical exponent θ_c varies as c varies. More precisely, the observed c-dependence of θ_c supports the conjecture that

$$\theta_c = \theta_0 - c \qquad (c = 0, 1, 2, \ldots). \tag{1.7}$$

The implication of (1.3) and (1.7) is that although trees and (unrestricted) animals are in the same universality class, c-animals ($c = 1, 2, 3, \ldots$) are all in different universality classes. At first sight, this conclusion does not seem to support the field theory arguments which suggest that the exponent is independent of cycle fugacity. However, Whittington *et al* (1983) have presented an heuristic argument—a crucial element of which is the conjectured c-dependence of θ_c—which shows how their results may be reconciled with the field theory prediction.

We have presented the above introduction to existing results in terms of weakly embeddable (or bond) clusters since, in this case, some of the steps can be proven rigorously. However, numerical evidence and general universality considerations indicate that precisely analogous results hold in the case of strongly embeddable (or site) clusters (Gaunt *et al* 1982, Whittington *et al* 1983).

This paper is concerned with the growth constants and the critical exponents associated with the total numbers of lattice trees with n sites *and specified topologies*. Of particular interest will be the way in which the critical exponent depends—if at all—on parameters associated with the specific topology, e.g. the number (b) of branches, the number (n_i) of branching points of degree i ($i = 3, 4, \ldots, q$), the number (n_1) of 'dangling ends' (i.e. vertices of degree 1), etc. The simplest type of tree is the chain with the topology shown in figure 1(a); more complicated topologies are shown

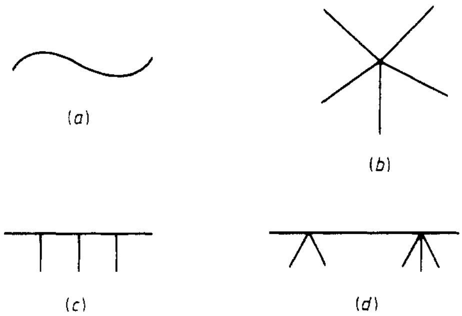

Figure 1. Examples of simple topologies: (a) simple chain, (b) star, (c) comb, (d) brush.

in figures 1(b), (c) and (d). (Throughout figure 1, all vertices of degree 2 have been suppressed since they do not affect the topology or consequently the structure of the branched polymer.)

The topology of the simple chain (see figure 1(a)) clearly has a single branch ($b = 1$) and two dangling ends ($n_1 = 2$). To deal with the more complicated topologies in figure 1, we define—quite generally—a *branch* of the topology as a segment of simple chain terminated at each end by either a dangling end or a branching point. For the topologies shown in figure 1, $b = 1$, 5, 7 and 8, respectively. Using Euler's law of the edges it follows that

$$b = n^+ - 1 + n_1, \tag{1.8}$$

where $n^+ = \Sigma_{i=3}^q n_i$ is the total number of vertices with degree greater than two, i.e. the total number of branching points. Alternatively, we may use Euler's law to show that

$$n_1 = 2 + \sum_{i=3}^{q} (i-2) n_i \tag{1.9}$$

and, hence, that both n_1 and

$$b = 1 + \sum_{i=3}^{q} (i-1) n_i \tag{1.10}$$

are determined by the integer set $(n_3, n_4, \ldots, n_q)$.

The statistics of simple chains (see figure 1(a)) are already well known, see for example the reviews by McKenzie (1976) and Whittington (1982). If c_n is the number of weakly embeddable chains with n sites, then asymptotically one writes

$$c_n \sim n^{\gamma-1} \mu^n \qquad (n \to \infty) \tag{1.11}$$

where μ is the self-avoiding walk (SAW) limit and γ is the associated critical exponent. For two-dimensional lattices, we have the presumably exact value (Nienhuis 1982) of

$$\gamma = 1\tfrac{11}{32} = 1.34375 \qquad (d = 2) \tag{1.12}$$

while in three dimensions the most precise estimate of γ, namely

$$\gamma = 1.1615 \pm 0.0020 \qquad (d = 3), \tag{1.13}$$

has been obtained from renormalisation group field-theoretic calculations (Baker *et al* 1978, Le Guillou and Zinn-Justin 1980). Estimates of γ from exact enumeration techniques (Watts 1975, and references therein) are very close to the values given in (1.12) and (1.13). For the SAW problem, the classical value of γ is $\gamma = 1$ and the upper critical dimension is $d_c = 4$, which implies that

$$\gamma = 1 \qquad (d \geq 4). \tag{1.14}$$

The topologies in figures 1(b) and (c) are examples of 'stars' and 'combs', respectively (de Gennes 1979). More generally, let us define $s(n; b)$ to be the number of weakly embeddable stars with a total of n sites and b branches, and denote by $c(n; t)$ the number of weakly embeddable combs with n sites and t 'teeth'. We have called the topology in figure 1(d) a 'brush' and, more generally, will use $b(n; m_1, \ldots, m_t)$ to denote the number of weak embeddings with n sites and t 'bristles' of multiplicities $m_1, m_2, \ldots, m_t$, respectively. For combs and brushes, the number of branches is given by

$$b = \sum_{i=1}^{t} m_i + t + 1. \tag{1.15}$$

(Note that $s(n; 3) \equiv c(n; 1)$ and $s(n; b') \equiv b(n; b' - 2)$ for $b' \geq 3$.) The topology in figure 1(c) may be regarded as either a comb with $t = 3$ teeth or as a brush with $t = 3$ bristles with multiplicities $m_1 = m_2 = m_3 = 1$. The brush in figure 1(d) has $t = 2$ bristles with $m_1 = 2$, $m_2 = 3$.

For complicated topologies, the acquisition of exact enumeration data becomes rapidly more difficult. Consequently, all the topologies that we study in detail in § 3 are *simple* examples of either stars, combs or brushes. For all these topologies, it has been proved (Lipson and Whittington 1983) that the growth constants exist and are all equal to the SAW limit μ. More precisely, Lipson and Whittington (1983) have proved that

$$\lim_{n \to \infty} n^{-1} \log t(n; n_3, n_4, \ldots, n_q) = \log \mu, \tag{1.16}$$

where $t(n; n_3, n_4, \ldots, n_q)$ is the number, per lattice site, of trees with n vertices and a specified number n_i of vertices of degree i ($i = 3, 4, \ldots, q$), weakly embeddable in a d-dimensional hypercubic lattice. Note that knowledge of $n, n_3, n_4, \ldots$ is sufficient to determine not only n_1 from (1.9) but also n_2 through

$$n_2 = n - n_1 - n^+ = n - 2 - \sum_{i=3}^{q} (i-1) n_i. \tag{1.17}$$

The rigorous result in (1.16) proves extremely useful for the series analysis which we perform in § 3 ($d = 2, 3$) and §4 ($d = 4$) on the exact enumeration data and enables us to focus attention on the associated critical exponents. Assuming the expected asymptotic form

$$t(n; n_3, n_4, \ldots, n_q) \sim n^{\gamma_t - 1} \mu^n \qquad (n \to \infty) \tag{1.18}$$

we estimate γ_t and find that the data are consistent with

$$\gamma_t = \gamma + b - 1 \tag{1.19}$$

in all the cases studied.

It should be noted, however, that values of $n_3, n_4, \ldots, n_q$ do not necessarily specify a unique topology. For example, both of the topologies in figure 2 have $n_3 = 4$, $n_4 = n_5 = \ldots = 0$ and yet are distinct. It follows that although (1.16) is sufficient to

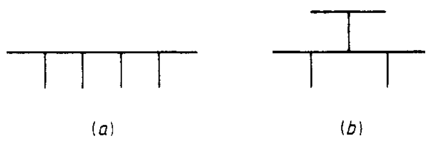

Figure 2. Distinct topologies both having four vertices of degree 3 and none of higher degree.

prove that the growth constant is μ for topologies as simple as the ones we study in § 3, it does not establish this result for an arbitrary topology. Nevertheless, we *expect* the growth constant to be μ for *all* topologies. Some relevant rigorous results are presented in § 2. In addition, we *expect* each of the distinct topologies associated with a given set of integers $(n_3, n_4, \ldots, n_q)$ to have the same critical exponent, (1.19), as their union. (This is because all topologies associated with the set $(n_3, n_4, \ldots, n_q)$ must have, from (1.10), the same number of branches.)

The result (1.19) *is a conjecture.* Besides being consistent with the exact enumeration data (see §§ 3 and 4 and table 2), it of course agrees with (1.11) in the case of simple chains ($b = 1$), it satisfies the exact upper bounds derived in § 2 (see also table 2) and is the correct result for the Bethe approximation (see § 4). In addition, we give in its support two different arguments, both heuristic in nature. The first of these depends upon a result which we prove rigorously in § 2. Consider a 'realisation' of a specific topology and allow the length of one of its branches to increase indefinitely while the lengths of the other branches remain fixed. Then, according to a theorem proved in § 2, the growth constant for this sequence of realisations of the given topology is μ. Furthermore, we expect the critical exponent to be chain-like since the large n behaviour should be dominated by the branch whose length is allowed to grow indefinitely and be essentially unaffected by the branches of fixed lengths. This gives a factor of $n^{\gamma-1}\mu^n$ for this sequence of realisations. The *number* of realisations of any given topology increases like n^{b-1} as $n \to \infty$ (Gupta *et al* 1958, Domb and Heap 1967), and so overall we might reasonably expect asymptotic behaviour of the form $n^{b-1}n^{\gamma-1}\mu^n$. Not only does this simple argument give a growth factor of μ, in agreement with the rigorous result (1.16), but comparison with (1.18) yields the conjectured form, (1.19), for γ_t. A hidden assumption in the above argument is that equal amplitudes are associated with each of the sequences of realisations. In fact, numerical analysis (unpublished work) of several sequences suggests more complicated behaviour and the above treatment, therefore, implicitly assumes a constant 'effective amplitude'.

The second of the heuristic arguments to support (1.19) is presented in § 5 and is analogous to the one given by Whittington *et al* (1983) in support of their conjecture (1.7). The idea is to show how the conjecture (1.19) and the rigorous result (1.16) are crucial ingredients in an heuristic theory designed to demonstrate that the universality class for branched polymers without cycles is independent of the branching fugacity $z > 0$. The introduction of branching fugacities is a natural device in the field-theoretic treatment of branched polymers (Lubensky and Isaacson 1979) and our conclusion that the universality class is independent of branching fugacity was anticipated in the position space renormalisation group calculations of Family (1980).

In § 3 the problem of strongly embeddable (or site) lattice trees with a specified topology is also studied. (We use the upper case letters $S(n; b)$, $C(n; t)$ and $B(n; m_1, m_2, \ldots, m_t)$ to denote the numbers of strongly embeddable stars, combs and brushes, respectively.) We assume, in analogy with (1.18), an asymptotic form

$$T(n; n_3, n_4, \ldots, n_q) \sim n^{\gamma_T - 1} \upsilon^n \qquad (n \to \infty) \tag{1.20}$$

since it follows rigorously from § 2 that the dominant behaviour is determined by the growth constant υ for neighbour-avoiding walks (Whittington 1982), i.e. strongly embeddable SAWs. From an analysis of the exact enumeration data, we estimate γ_T and find evidence supporting the universality conjecture

$$\gamma_T = \gamma_t. \tag{1.21}$$

In § 4 we give the numbers of undirected neighbour-avoiding walks on a d-dimensional simple hypercubic lattice, for arbitrary integral d and up to $n = 9$ sites. We use these expressions to derive an expansion for υ in inverse powers of σ $(= 2d - 1)$ correct through order σ^{-3}. We follow a similar procedure for strongly embeddable stars with three branches and up to $n = 9$ sites. In this case, the $1/\sigma$ expansion for the growth constant is derived through order σ^{-2} and agrees term by term with the corresponding expansion for υ. This finding is consistent with the rigorous results of § 2. Similar results have been obtained for weakly embeddable trees with a given topology.

Finally, in § 6, our results are summarised and discussed.

2. Invariance of growth constants and bounds on exponents for simple branched trees

In this section we consider the set of trees with n vertices, n_3 of degree 3, n_4 of degree 4, etc, strongly embeddable in the d-dimensional hypercubic lattice. We show that, for a certain subset of these trees, allowing the number of edges in one of the simple chains to go to infinity yields a growth constant which is the same as the corresponding growth constant for neighbour-avoiding walks. Similar arguments can be constructed relating the growth constants of weakly embeddable trees and self-avoiding walks. We then proceed to derive bounds on the corresponding critical exponents for some simple topologies.

We first need some results on neighbour-avoiding walks. Let C_n be the number of undirected neighbour-avoiding walks with n vertices. By arguments exactly analogous to those of Hammersley and Morton (1954) for SAWs, it is easy to show that there exists a finite positive constant υ such that

$$\lim_{n \to \infty} n^{-1} \log C_n = \inf_{n > 0} n^{-1} \log C_n = \log \upsilon. \tag{2.1}$$

Consider the subset $(U_n(k))$ of undirected neighbour-avoiding walks with n vertices whose two vertices of unit degree are the sole members of vertex subsets having largest and smallest values respectively of some specified coordinate, k. Using an unfolding transformation analogous to that used by Hammersley and Welsh (1962) it is easy to show that the number, B_n, of such walks satisfies

$$\lim_{n \to \infty} n^{-1} \log B_n = \sup_{n > 0} n^{-1} \log B_n = \log \upsilon, \tag{2.2}$$

i.e., the two sets of walks have the same growth constant.

We represent a particular topological class of trees (e.g. one of the classes shown in figure 1) by a homeomorphically irreducible graph G. Choose a pair of vertices v_a and v_b which are adjacent in G. In other words v_a and v_b are joined by a simple chain in a realisation of the tree. Removing the edge joining v_a and v_b in G decomposes the graph into two connected components G_a and G_b which contain v_a and v_b respectively. Suppose that it is possible to find two integers n_a and n_b such that one can construct strongly embeddable realisations of G_a and G_b, containing n_a and n_b vertices respectively. Suppose in addition that for some k, v_a belongs to the vertex subset with largest k-coordinate in the embedding of G_a and v_b belongs to the vertex subset with smallest k-coordinate in the embedding of G_b. For an example relevant to the following argument see figure 3. Translate the embedding of G_a so that v_a is at the origin. For each $u \in U_m(k)$ translate u so that the unit degree vertex with smallest k-coordinate is at $x_k = 1$, $x_j = 0$, $\forall j \neq k$. Suppose that the other unit degree vertex of u has coordinates $(x_1^0, x_2^0, \ldots, x_k^0, \ldots)$. Translate the embedding of G_b such that the coordinates of v_b are $(x_1^0, x_2^0, \ldots, x_k^0 + 1, \ldots)$. By adding the two appropriate edges to join the unit degree vertices of u to v_a and v_b we have constructed a realisation of G with $(n_a + n_b + m)$ vertices which is strongly embeddable in the lattice. Since there are B_m such simple chains and, by hypothesis, at least one strong embedding of the realisations of G_a and G_b, the number of strong embeddings with n vertices, $T_G(n)$, of the topological class represented by the graph G is bounded below as

$$T_G(n) \geq B_{n-n_a-n_b}. \tag{2.3}$$

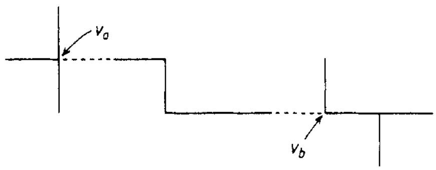

Figure 3. Example of a construction of G from G_a, G_b and a neighbour-avoiding walk.

Taking logarithms, fixing n_a and n_b and letting n tend to infinity, (2.2) and (2.3) imply that

$$\liminf_{n\to\infty} n^{-1} \log T_G(n) \geq \log \upsilon. \tag{2.4}$$

To construct an upper bound we note that the arguments given by Lipson and Whittington (1983) for an upper bound on weak embeddings with specified $n_3, n_4, \ldots$ can be taken over *mutatis mutandis* to this problem, with SAWs replaced by neighbour-avoiding walks. This argument shows that

$$\limsup_{n\to\infty} n^{-1} \log T_G(n) \leq \log \upsilon. \tag{2.5}$$

Then (2.4) and (2.5) imply

$$\lim_{n\to\infty} n^{-1} \log T_G(n) = \log \upsilon. \tag{2.6}$$

To apply this general result to some of the topologies with which we are especially concerned in this paper, we note that choosing G_a to be a single vertex and G_b to be a star with s vertices, $s-1$ of degree 1 and one of degree $s-1$ (the degree $(s-1)$

vertex being labelled v_b), it follows that stars with s branches have growth constant υ. Similarly if we choose G_a to be a star with s' vertices, $s'-1$ of degree 1 and one of degree $s'-1$ (with the vertex of degree $s'-1$ labelled v_a), then concatenating this G_a with the above G_b through each member of a set of appropriate simple chains shows that this kind of brush with two vertices of degree s and s' has growth constant υ. Similar arguments can be constructed for other topologies.

We have focused on the specific case in which one simple chain in the graph becomes infinite; these arguments also establish the invariance of the growth constant when any number of simple chains in the graph grow to infinity.

We now consider bounds on the associated critical exponents (defined for instance through (1.18)). The arguments given will be for strong embeddings but analogous reasoning can be applied to the case of weak embeddings.

The general approach for deriving upper bounds will be to construct the graph from a set of neighbour-avoiding walks; since these walks are not necessarily mutually avoiding this will yield an upper bound on the appropriate number of embeddings.

Consider a star with b branches. If b is even this star can be constructed from $b/2$ neighbour-avoiding walks, which intersect at a common vertex (which is not of unit degree for any walk). The number of embeddings of the b-star with n vertices is then bounded above by

$$S(n;b) \leqslant \sum_{m_1}\sum_{m_2}\dots\sum_{m_{b/2}} C_{m_1}C_{m_2}\dots C_{m_{b/2}}(m_1-2)(m_2-2)\dots(m_{b/2}-2) \tag{2.7}$$

where

$$m_l \geqslant 3 \qquad \forall l \tag{2.8}$$

and

$$m_1+m_2+\dots+m_{b/2}=n+\tfrac{1}{2}b-1. \tag{2.9}$$

The factors (m_1-2) etc arise from the number of ways of choosing the common vertex in each walk. Assuming the usual asymptotic form

$$C_n \sim n^{\gamma-1}\upsilon^n \tag{2.10}$$

where γ is thought to be the same for self-avoiding and neighbour-avoiding walks (Watson 1970),

$$\begin{aligned}\sum_{m_1}\sum_{m_2}\dots\sum_{m_{b/2}} & C_{m_1}C_{m_2}\dots C_{m_{b/2}}(m_1-2)(m_2-2)\dots(m_{b/2}-2)\\ &\sim \sum_{m_1}\sum_{m_2}\dots\sum_{m_{b/2}}(m_1m_2\dots m_{b/2})^{\gamma}\upsilon^n\\ &\leqslant An^{(b/2)-1}n^{(b/2)\gamma}\upsilon^n\\ &= An^{(b/2)(\gamma+1)-1}\upsilon^n.\end{aligned} \tag{2.11}$$

Hence, if

$$S(n;b)\sim n^{\gamma_{S,b}-1}\upsilon^n,$$

(2.8) and (2.11) give

$$\gamma_{S,b}\leqslant\tfrac{1}{2}b(\gamma+1), \qquad b \text{ even}. \tag{2.12}$$

A similar argument for b odd yields

$$\gamma_{S,b}\leqslant\tfrac{1}{2}b(\gamma+1)+\tfrac{1}{2}(\gamma-1), \qquad b \text{ odd}. \tag{2.13}$$

For other cases, such as a brush having t bristles with multiplicities $m_1, m_2, \ldots,$ we proceed in an analogous manner. The number of branches, b, will be given by (1.15) and there will be $O(n^{b-1})$ ways of distributing the edges among the different branches. The brush can be constructed from $1+\Sigma_i m_i$ neighbour-avoiding walks (the 'backbone' of the brush and the bristles). It is then easy to show that an upper bound on the exponent $\gamma_{B,m_1,m_2,\ldots}$ is given by

$$\gamma_{B,m_1,m_2,\ldots} - 1 \leq b - 1 + (\gamma - 1)\left(1 + \sum m_i\right) \qquad (2.14)$$

or

$$\gamma_{B,m_1,m_2\ldots} \leq \gamma(b-t) + t. \qquad (2.15)$$

In the special case of a comb this becomes

$$\gamma_{C,t} \leq \gamma(t+1) + t. \qquad (2.16)$$

For teeth with multiplicities greater than one the bound (2.15) can be improved by using one neighbour-avoiding walk to make up two bristles in a manner similar to that described above for stars.

It is also easy to show that the exponent for each of these topologies is bounded below by the SAW exponent γ.

It is interesting to note that for dimensionality $d \geq d_c$, when according to (1.14) we have $\gamma = 1$, the upper bounds (2.12), (2.13), (2.15) and (2.16) all coincide with our conjecture (1.19) which reduces to $\gamma_t = b$. More generally, for arbitrary d, these bounds imply that $\gamma_t = O(b)$, which rules out the possibility of powers of b greater than the first.

3. Series derivation and analysis: $d = 2$ and 3

We have derived exact enumeration data for stars, combs and brushes, both weakly and strongly embeddable in the square, triangular and simple cubic lattices. The data for weak embeddings are presented in appendix 1 and those for strong embeddings in appendix 2. (Note that in the appendices and throughout this section we use n to denote the number of bonds (sites) for weak (strong) embeddings. Although we have found it more convenient to use different conventions in other sections, no confusion should arise in practice.) Data for the stars $s(n; 3)$ weakly embeddable in the triangular, diamond, simple cubic, body-centred cubic and face-centred cubic lattices have been given by McKenzie (1967) through orders $n = 10, 11, 12, 12$ and 10 bonds, respectively. With this exception the data in appendices 1 and 2 appear to be new. We reproduce McKenzie's data, which are somewhat inaccessible, for the diamond, body-centred cubic and face-centred cubic lattices in appendix 3.

For both weak and strong embeddings, we have considered stars with $b = 3$, 4, 5 and 6 branches, combs with $t = 2$ and 3 teeth, and brushes with $t = 2$ bristles either with multiplicities one and two or both of multiplicity two. For strong embeddings, the data extend through $n = 16$, 13 and 11 sites for the square, triangular and simple cubic lattices, respectively. (Note, however, that for the triangular lattice $S(n; b) = 0$ for all n for $b = 4$, 5 and 6 and that there are no brushes, i.e. $B(n; m_1, m_2, \ldots, m_t) = 0$ for all possible n and $\{m_i\}$.) For weak embeddings, the extent of our data depends not only on the lattice but also on the topology; in the most favourable cases the data are through $n = 16$, 14 and 14 bonds for the square, triangular and simple cubic lattices, respectively.

For the strong embeddings, the data were derived by first enumerating (by computer) *all* lattice trees with n sites ($n = 1, 2, 3, \ldots$) and then classifying them according to the integer set $(n_3, n_4, \ldots, n_q)$. As we saw in § 1, simple topologies are specified uniquely by such a set. A similar procedure was used to derive the data for weak embeddings through $n = 14$, 10 and 10 bonds for the square, triangular and simple cubic lattices, respectively. Larger values of n are impracticable by this technique since they are too expensive in computer time. However, in the case of the weak embeddings, some additional data have been derived by computer enumeration of all possible realisations, for a given n, of a specified topology. The numbers of possible realisations for various topologies and values of n are given in table 1. (As mentioned in § 1, these numbers are known to increase asymptotically like n^{b-1} for all topologies.) This technique too is soon limited by the available computing time.

Table 1. Numbers of possible realisations for various topologies and $n \leq 20$.

n	$s(n;3)$	$s(n;4)$	$s(n;5)$	$s(n;6)$	$c(n;2)$	$c(n;3)$	$b(n;1,2)$	$b(n;2,2)$
3	1							
4	1	1						
5	2	1	1		1			
6	3	2	1	1	2		1	
7	4	3	2	1	5	1	3	1
8	5	5	3	2	9	3	8	2
9	7	6	5	3	17	10	17	5
10	8	9	7	5	27	24	33	10
11	10	11	10	7	43	55	58	20
12	12	15	13	11	63	109	97	35
13	14	18	18	14	92	206	153	61
14	16	23	23	20	127	360	233	98
15	19	27	30	26	174	606	342	155
16	21	34	37	35	230	970	489	234
17	24	39	47	44	302	1508	681	347
18	27	47	57	58	386	2264	930	498
19	30	54	70	71	490	3322	1245	705
20	33	64	84	90	610	4750	1641	973

We have analysed all the data given in the appendices using standard series analysis methods (Gaunt and Guttmann 1974). For example, for the star $s(n;3)$ weakly embeddable in the square lattice we have plotted against $1/n$ in figure 4 the ratios $r_n = s(n;3)/s(n-1;3)$, the linear extrapolants $r'_n = \frac{1}{2}[nr_n - (n-2)r_{n-2}]$ from alternate points and their averages $r''_n = \frac{1}{2}(r_n + r'_n)$. According to (1.16), all these plots should approach μ as $n \to \infty$. The arrow in figure 4 indicates the unbiased estimate of μ given by Watts (1975). The corresponding evidence for the triangular, diamond, simple cubic, body-centred cubic and face-centred cubic lattices is equally satisfactory. Note that for the close-packed lattices, the extrapolants r'_n have been calculated from adjacent points, i.e. $r'_n = nr_n - (n-1)r_{n-1}$.

The exponent $\gamma_{s,3}$ can be estimated from the sequence of biased estimates

$$\gamma_{s,3}(n) = 1 + n[(r_n/\hat{\mu}) - 1]$$

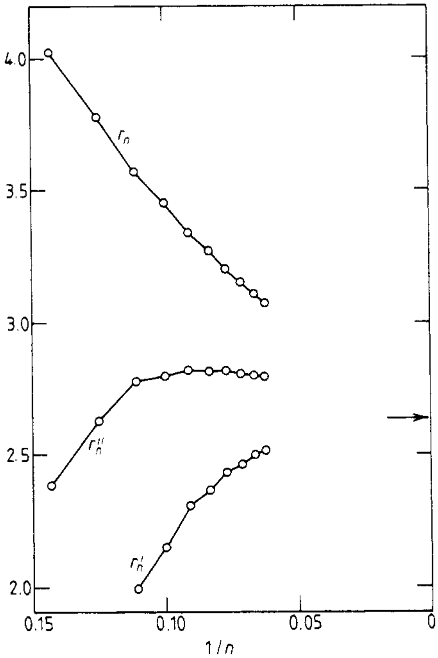

Figure 4. Plots against $1/n$ of unbiased ratio estimates of the growth parameter for weak embeddings of the star $s(n; 3)$ on the square lattice. The arrow indicates an unbiased estimate of μ (Watts 1975).

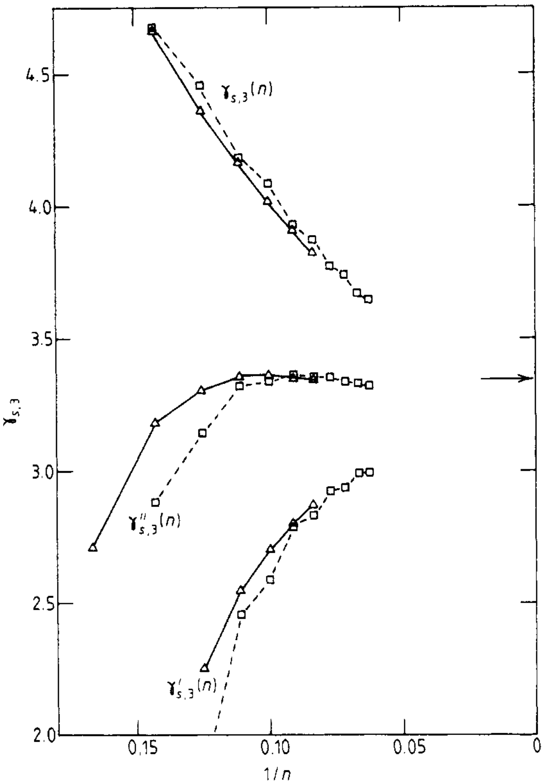

Figure 5. Plots against $1/n$ of biased ratio estimates of $\gamma_{s,3}$ for weak embeddings of the star $s(n; 3)$ on the square (□) and triangular (△) lattices. The conjectured exponent $(\gamma+2)$ is indicated by an arrow.

where $\hat{\mu}$ is an estimate of μ for which we have used the unbiased estimate of Watts (1975). We also form linear extrapolants

$$\gamma'_{s,3}(n) = [n\gamma_{s,3}(n) - (n-m)\gamma_{s,3}(n-m)]/m$$

from alternate points $(m=2)$ for loose-packed lattices or adjacent points $(m=1)$ for close-packed lattices and the averages

$$\gamma''_{s,3}(n) = \tfrac{1}{2}[\gamma_{s,3}(n) + \gamma'_{s,3}(n)].$$

Plots of $\gamma_{s,3}(n)$, $\gamma'_{s,3}(n)$ and $\gamma''_{s,3}(n)$ against $1/n$ are given in figure 5 for the square and triangular lattices. Similar plots are obtained in three dimensions. Our best estimates of $\gamma_{s,3}$ are

$$\gamma_{s,3} = 3.2 \pm 0.25 \qquad (d=2) \tag{3.1}$$

$$= 3.15 \pm 0.30 \qquad (d=3). \tag{3.2}$$

We have analysed, using numerical techniques identical to those outlined above, our data for combs, brushes and the remaining stars with $b=4$, 5 and 6 branches. Some typical plots are shown in figures 6 and 7. In figure 6 we plot, against $1/n$, estimates of $\gamma_{c,2}$ for combs with $t=2$ teeth on the square and triangular lattices.

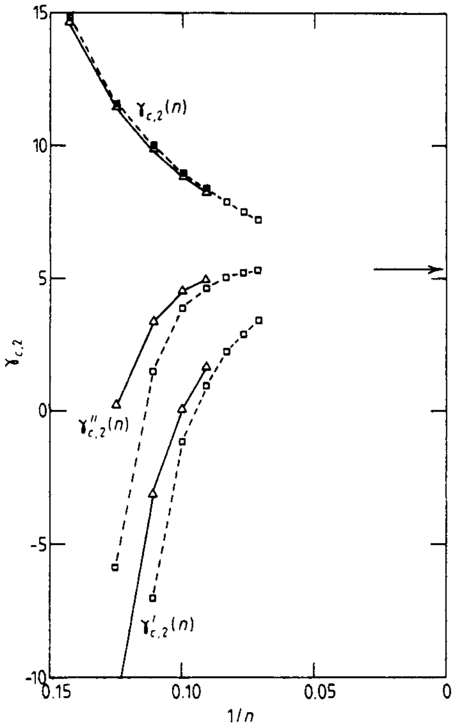

Figure 6. Plots against $1/n$ of biased ratio estimates of $\gamma_{c,2}$ for weak embeddings of the comb $c(n;2)$ on the square (□) and triangular (△) lattices. The conjectured exponent $(\gamma+4)$ is indicated by an arrow.

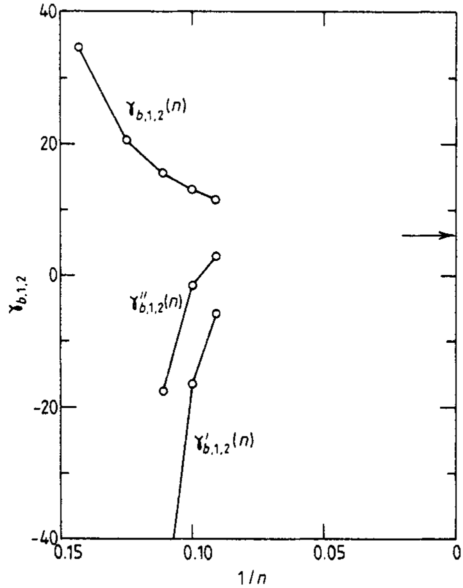

Figure 7. Plots against $1/n$ of biased ratio estimates of $\gamma_{b,1,2}$ for weak embeddings of the brush $b(n;1,2)$ on the simple cubic lattice. The conjectured exponent $(\gamma+5)$ is indicated by an arrow.

Estimates of $\gamma_{b,1,2}$ for brushes with $t=2$ bristles of multiplicities $m_1=1$ and $m_2=2$ on the simple cubic lattice are plotted against $1/n$ in figure 7. In figures 5, 6 and 7, the exponent $(\gamma+b-1)$, as conjectured in (1.19), is indicated by an arrow and the uncertainties induced by the uncertainties in μ are less than the size of the symbols.

Our best estimates of all the critical exponents studied are presented in table 2 where they are compared with our conjectured exponent $(\gamma+b-1)$. We also give in table 2 numerical values for the exact upper bounds derived in § 2. From this table (see also figures 5, 6 and 7), we see that our results are not inconsistent with the conjecture, *although other possibilities cannot be ruled out.* However, it appears that the possibility of the exact inequalities (2.12) and (2.13) for stars holding as strict equalities is probably excluded by our results for stars with $b=3$ and $b=4$ branches in two dimensions.

It is not difficult to understand why the quality of our results is, in general, relatively poor. For a given topology, the most numerous realisations and those which will dominate the asymptotic behaviour are those in which all the branches are of different lengths. For a topology with b branches, one requires at least $\frac{1}{2}b(b+1)$ bonds before such a realisation can occur. The only cases where our series are at least this extensive are stars with $b=3$ and $b=4$ branches. It is in just these cases that our exponent estimates are the most precise and hence provide the strongest support for our conjecture (1.19) while apparently excluding, with some degree of confidence, our exact upper bounds for stars holding as strict equalities.

Table 2. Series estimates of critical exponents for $d = 2$ and 3 dimensions. The conjectured exponent $(\gamma + b - 1)$ and the exact upper bounds are given for comparison purposes.

	$d=2$				$d=3$			
			Series estimates				Series estimates	
Topology	Exact upper bound	$\gamma+b-1$	Weak	Strong	Exact upper bound	$\gamma+b-1$	Weak	Strong
Star $b=3$	3.6875	3.343 75	3.2 ± 0.25	3.3 ± 0.25	3.323	3.1615	3.15 ± 0.3	3.2 ± 0.6
Star $b=4$	4.6875	4.343 75	4.1 ± 0.6	4.0 ± 0.5	4.323	4.1615	4.0 ± 1.0	3.75 ± 1.25
Star $b=5$	6.031 25	5.343 75	5.0 ± 1.5	—	5.4845	5.1615	5.0 ± 1.5	4.25 ± 1.75
Star $b=6$	7.031 25	6.343 75	5.5 ± 2.0	—	6.4845	6.1615	6.0 ± 3.0	---
Comb $t=2$	6.031 25	5.343 75	5.5 ± 2.0	5.5 ± 2.0	5.4845	5.1615	5.0 ± 3.0	4.0 ± 3.0
Comb $t=3$	8.375	7.343 75	7.0 ± 5.0	7.0 ± 5.0	7.646	7.1615	---	---
Brush $m_1=1$, $m_2=2$	7.375	6.343 75	6.0 ± 3.0	6.0 ± 3.0	6.646	6.1615	6.0 ± 4.0	---
Brush $m_1=2$, $m_2=2$	8.718 75	7.343 75	7.0 ± 4.0	7.0 ± 4.0	7.8075	7.1615	---	---

--- No estimate possible with available data.

— No embeddings possible of such topologies.

We have analysed the corresponding data for strong embeddings in exactly the same way. Provided that lim r_n exists, it readily follows from the lemma proved in § 2 that the ratios r_n, their extrapolants r'_n and the averages r''_n should all approach υ as $n\to\infty$ for all the topologies that we have studied. As an example, we have plotted these quantities against $1/n$ in figure 8 for the comb $C(n; 2)$ on the triangular lattice. The arrow indicates the best available estimate of υ (Whittington *et al* 1979). (Estimates of υ for the square and simple cubic lattices are given by Gaunt *et al* 1979, 1980).) The corresponding plots for other lattices and topologies are equally satisfactory in most cases.

The ratio analysis plots for the critical exponents are numerous and rather similar to those obtained for the weak embeddings. They are therefore omitted and our best estimates of the critical exponents are given in table 2, where they may be compared with the corresponding estimates for weak embeddings and with the conjectured value $(\gamma+b-1)$. Clearly, our results are consistent with the conjecture and support the hypothesis, expressed more generally in (1.21), of universal critical exponents for both weak and strong embeddings.

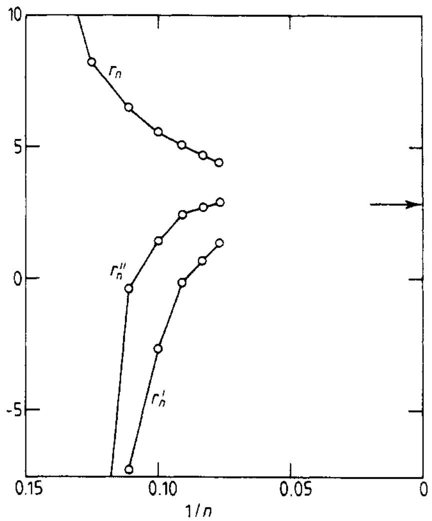

Figure 8. Plots against $1/n$ of unbiased ratio estimates of the growth parameter for strong embeddings of the comb $C(n; 2)$ on the triangular lattice. The arrow indicates the best available estimate of υ (Whittington *et al* 1979).

4. Expansions for hypercubical lattices

For the general d-dimensional simple hypercubic lattice, we have derived expressions for the numbers of strongly embeddable simple chains (or undirected neighbour-avoiding walks) with up to $n=8$ bonds. They are

$$C_1=\binom{d}{1}$$

$$C_2=\binom{d}{1}+4\binom{d}{2}$$

$$C_3=\binom{d}{1}+12\binom{d}{2}+24\binom{d}{3}$$

$$C_4=\binom{d}{1}+32\binom{d}{2}+168\binom{d}{3}+192\binom{d}{4}$$

$$C_5=\binom{d}{1}+80\binom{d}{2}+864\binom{d}{3}+2496\binom{d}{4}+1920\binom{d}{5}$$

$$C_6=\binom{d}{1}+196\binom{d}{2}+4032\binom{d}{3}+21\,888\binom{d}{4}+40\,320\binom{d}{5}+23\,040\binom{d}{6}$$

$$C_7=\binom{d}{1}+468\binom{d}{2}+17\,664\binom{d}{3}+163\,008\binom{d}{4}+541\,440\binom{d}{5}+714\,240\binom{d}{6}$$
$$+322\,560\binom{d}{7}$$

$$C_8=\binom{d}{1}+1120\binom{d}{2}+75\,624\binom{d}{3}+1121\,664\binom{d}{4}+5955\,840\binom{d}{5}+13\,639\,680\binom{d}{6}$$
$$+13\,870\,080\binom{d}{7}+5160\,960\binom{d}{8} \tag{4.1}$$

where $\binom{d}{r}$ are binomial coefficients. To calculate the coefficients of the binomial coefficients, we first computed C_n ($n=1,2,3,\dots$) for fixed d ($=4,5,\dots$) by machine enumeration. Data for the cases $d=2$ and 3 are already known through orders $n=24$ and 16, respectively (Whittington *et al* 1979, Gaunt *et al* 1980).

We now use these results to derive an expression for υ in inverse powers of $\sigma=2d-1$. The data in (4.1) may be written in the general form

$$C_n(d)=\sum_{\xi=0}^{n-1} C_n^{\xi}\binom{d}{n-\xi}. \tag{4.2}$$

For $\xi=0$, 1, 2 and 3 we have been able to calculate C_n^{ξ} as functions of n,

$$C_n(d)=2^{n-1}n!\binom{d}{n}+2^{n-2}(n-1)!(n^2-3n+3)\binom{d}{n-1}$$
$$+2^{n-3}(n-2)!\tfrac{1}{6}(3n^4-26n^3+87n^2-136n+96)\binom{d}{n-2}$$
$$+2^{n-4}(n-3)!\tfrac{1}{6}(n^6-17n^5+120n^4-453n^3+983n^2$$
$$-1228n+852)\binom{d}{n-3}+\dots+\binom{d}{1} \qquad (n\geq 6). \tag{4.3}$$

Following the approach outlined by Gaunt *et al* (1976) we expand the binomial coefficients in (4.3) in inverse powers of σ giving

$$C_n(d)=\tfrac{1}{2}\sigma^n[1-(n-3)\sigma^{-1}+\tfrac{1}{2}(n^2-7n+14)\sigma^{-2}-\tfrac{1}{6}(n^3-12n^2+71n-246)\sigma^{-3}+\dots]. \tag{4.4}$$

Hence,

$$\ln\upsilon(d)=\lim_{n\to\infty} n^{-1}\ln C_n(d)=\ln\sigma-\sigma^{-1}-\tfrac{1}{2}\sigma^{-2}-3\tfrac{1}{3}\sigma^{-3}-\dots \tag{4.5}$$

or, taking exponentials,

$$\upsilon=\sigma(1-\sigma^{-1}-3\sigma^{-3}-\dots). \tag{4.6}$$

The prefactor term in (4.6) corresponds, as expected, to the Bethe approximation $\upsilon=\sigma$ and the leading correction term is of first order in $1/\sigma$.

In a similar way, we have derived the following expressions for the numbers $S(n;3)$ of stars with three branches and up to $n=8$ bonds strongly embeddable in a d-dimensional hypercubic lattice:

$$S(3;3)=4\binom{d}{2}+8\binom{d}{3}$$

$$S(4;3)=20\binom{d}{2}+144\binom{d}{3}+192\binom{d}{4}$$

$$S(5;3)=84\binom{d}{2}+1320\binom{d}{3}+4416\binom{d}{4}+3840\binom{d}{5}$$

$$S(6;3)=308\binom{d}{2}+9264\binom{d}{3}+59\,584\binom{d}{4}+122\,880\binom{d}{5}+76\,800\binom{d}{6}$$

$$S(7;3)=1048\binom{d}{2}+57\,696\binom{d}{3}+631\,872\binom{d}{4}+2328\,960\binom{d}{5}$$
$$+3340\,800\binom{d}{6}+1612\,800\binom{d}{7}$$

$$S(8;3)=3312\binom{d}{2}+329\,400\binom{d}{3}+5821\,824\binom{d}{4}+34\,379\,520\binom{d}{5}$$
$$+85\,178\,880\binom{d}{6}+92\,252\,160\binom{d}{7}+36\,126\,720\binom{d}{8}. \tag{4.7}$$

It can be shown more generally that

$$S(n;3)=\sum_{\xi=0}^{n-2}S_n^{\xi}\binom{d}{n-\xi} \tag{4.8}$$

$$=\frac{1}{3!}2^{n-1}n!(n-1)(n-2)\binom{d}{n}+\frac{1}{3!}2^{n-2}(n-1)!(n-2)(n^3-4n^2+3n+6)$$
$$\times\binom{d}{n-1}+\frac{1}{3!}2^{n-3}(n-2)!\tfrac{1}{6}(3n^6-35n^5+153n^4-269n^3$$
$$+700n-1020)\binom{d}{n-2}+\dots \qquad (n\geqslant 4) \tag{4.9}$$

and hence that

$$\lim_{n\to\infty} n^{-1}\ln S(n;3)=\ln\sigma-\sigma^{-1}-\tfrac{1}{2}\sigma^{-2}-\dots, \tag{4.10}$$

which is term-by-term identical with the analogous expansion (4.5) for $\ln\upsilon$, at least to this order. This is consistent with the result

$$\lim_{n\to\infty} n^{-1}\ln S(n;3)=\lim_{n\to\infty} n^{-1}\ln C_n=\ln\upsilon, \tag{4.11}$$

which readily follows from the lemma proved in § 2.

We have performed similar calculations for weak embeddings of stars with three branches. Expressions for $s(n;3)$ for $n \leqslant 9$ bonds are

$$s(3;3)=4\binom{d}{2}+8\binom{d}{3}$$

$$s(4;3)=36\binom{d}{2}+192\binom{d}{3}+192\binom{d}{4}$$

$$s(5;3)=192\binom{d}{2}+2280\binom{d}{3}+5760\binom{d}{4}+3840\binom{d}{5}$$

$$s(6;3)=872\binom{d}{2}+20\,176\binom{d}{3}+96\,640\binom{d}{4}+153\,600\binom{d}{5}+76\,800\binom{d}{6}$$

$$s(7;3)=3508\binom{d}{2}+151\,416\binom{d}{3}+1235\,520\binom{d}{4}+3509\,760\binom{d}{5}+4032\,000\binom{d}{6}$$
$$+1612\,800\binom{d}{7}$$

$$s(8;3)=13\,252\binom{d}{2}+1035\,696\binom{d}{3}+13\,515\,264\binom{d}{4}+61\,063\,680\binom{d}{5}$$
$$+120\,844\,800\binom{d}{6}+108\,380\,160\binom{d}{7}+36\,126\,720\binom{d}{8}$$

$$s(9;3)=47\,320\binom{d}{2}+6632\,184\binom{d}{3}+133\,791\,104\binom{d}{4}+903\,610\,240\binom{d}{5}$$
$$+2739\,394\,560\binom{d}{6}+4130\,595\,840\binom{d}{7}+3034\,644\,480\binom{d}{8}$$
$$+867\,041\,280\binom{d}{9}. \tag{4.12}$$

More generally, we find

$$s(n;3)=\sum_{\xi=0}^{n-2} s_n^{\xi}\binom{d}{n-\xi} \tag{4.13}$$

and the first few terms are

$$s(n;3)=\frac{1}{3!}2^{n-1}n!(n-1)(n-2)\binom{d}{n}+\frac{1}{3!}2^{n-2}(n-1)!\,n(n-1)(n-2)^2\binom{d}{n-1}$$
$$+\frac{1}{3!}2^{n-3}(n-2)!\,(n-3)\tfrac{1}{6}(3n^5-20n^4+48n^3-53n^2+12n+100)\binom{d}{n-2}$$
$$+\frac{1}{3!}2^{n-4}(n-3)!\tfrac{1}{6}(n^8-17n^7+120n^6-459n^5+1029n^4-1230n^3-64n^2$$
$$+3596n-7152)\binom{d}{n-3}+\ldots \qquad (n\geqslant 5). \tag{4.14}$$

The coefficients of $\binom{d}{n}$ in (4.14) and (4.9) are identical since, for clusters which stretch into the maximum possible number of dimensions, there is no difference between weak and strong embeddings. From (4.14) we obtain

$$\lim_{n\to\infty} n^{-1} \ln s(n;3) = \ln \sigma - \sigma^{-2} - 2\sigma^{-3} - \ldots. \tag{4.15}$$

For the problem of weakly embeddable chains (or undirected self-avoiding walks), Fisher and Gaunt (1964) give the analogue of (4.5) as

$$\ln \mu(d) = \lim_{n\to\infty} n^{-1} \ln c_n(d) = \ln \sigma - \sigma^{-2} - 2\sigma^{-3} - 11\tfrac{1}{2}\sigma^{-4} - 64\sigma^{-5} - \ldots. \tag{4.16}$$

The expansions (4.15) and (4.16) are term-by-term identical at least through order σ^{-3}. This observation is consistent with the exact result in (1.16).

On taking the exponential of (4.16) we get (Fisher and Gaunt 1964)

$$\mu = \sigma(1 - \sigma^{-2} - 2\sigma^{-3} - 11\sigma^{-4} - 62\sigma^{-5} - \ldots). \tag{4.17}$$

Although $1/\sigma$ expansions are probably only asymptotic, comparison of (4.17) with (4.6) suggests that the rigorous result $\mu(d) \geqslant \upsilon(d)$ is in fact a *strict* inequality for all d. Numerical estimates of μ and υ support this conjecture for $d = 2$ and 3 (Watts 1975, Gaunt *et al* 1979, 1980).

More generally, let us consider weakly embeddable stars with n bonds and b branches. The Bethe approximation for lattices of coordination number q may be obtained by first embedding the homeomorphically irreducible graph in $\binom{q}{b}$ ways and then multiplying by a factor to account for the additional $(n-b)$ bonds, giving

$$s(n;b) = \binom{q}{b}(q-1)^{n-b}\binom{n-1}{b-1}. \tag{4.18}$$

A detailed study shows that for hypercubical lattices with $q = 2d$ the leading correction to (4.18) is of $O(d^{n-2})$. Hence, following some algebra, we find

$$s(n;b) = \frac{1}{b!}2^n n!\binom{n-1}{b-1}\binom{d}{n} + \frac{1}{b!}2^{n-2}(n-1)!(2n^2 - 4n - b^2 + 3b) \times \binom{n-1}{b-1}\binom{d}{n-1} + \ldots \qquad (n \geqslant b). \tag{4.19}$$

Note that on setting $b = 3$ we reproduce the first two terms in (4.14). Expanding the binomial coefficients in inverse powers of σ gives

$$s(n;b) = \frac{1}{b!}\binom{n-1}{b-1}\sigma^n[1 - \tfrac{1}{2}b(b-3)\sigma^{-1} + O(\sigma^{-2})] \tag{4.20}$$

and hence

$$\lim_{n\to\infty} n^{-1} \ln s(n;b) = \ln \sigma + O(\sigma^{-2}). \tag{4.21}$$

If the star is strongly embeddable in a hypercubic lattice the leading correction to (4.18) is then of $O(d^{n-1})$. Consequently, the Bethe approximation gives only the leading contribution correctly, namely

$$S(n;b) = \frac{1}{b!}2^n n!\binom{n-1}{b-1}\binom{d}{n} + \ldots \qquad (n \geqslant b), \tag{4.22}$$

giving

$$\lim_{n\to\infty} n^{-1} \ln S(n;b) = \ln \sigma + O(\sigma^{-1}). \tag{4.23}$$

For all b, there is term-by-term agreement between (4.21) and (4.16) and between (4.23) and (4.5). Such agreement is consistent with the exact result (1.16) for weak embeddings and that proved in § 2 for strong embeddings. Although these results are not particularly impressive, owing to the shortness of the series, the calculation of further terms in (4.19) and (4.22) is a highly non-trivial problem.

More generally still, it seems that in the Bethe approximation the number of embeddings g_n of a graph G with n bonds is

$$g_n = \binom{n-1}{b-1} R(G) q(q-1)^{n-1-\lambda_2-\lambda_3-\ldots-\lambda_\sigma} \prod_{i=2}^{\sigma} (q-i)^{\lambda_i}. \tag{4.24}$$

Here $\sigma = q-1$ where q is the coordination number of the lattice, $R(G)$ is the reciprocal of the symmetry number of G and $\lambda_2, \lambda_3, \ldots, \lambda_\sigma$ are integers whose values depend on the topology under consideration. For example, setting $1/R(G) = b!$, $\lambda_i = 1$ for $2 \leq i \leq b-1$ and $\lambda_i = 0$ otherwise regains the result (4.18) for stars with b branches. The corresponding result for combs with n bonds and t teeth can be obtained by setting $R(G) = \frac{1}{8}$, $\lambda_i = t$ for $i = 2$ and zero otherwise. This gives

$$c(n; t) = \tfrac{1}{8} q(q-1)^{n-t-1}(q-2)^t \binom{n-1}{2t}, \tag{4.25}$$

where we have used $b = 2t+1$ which follows from (1.15) on setting $m_i = 1$ for all i. Similarly for brushes with t bristles of multiplicities $m_1, m_2, \ldots, m_t$, we set

$$1/R(g) = \Delta(m_1+1)(m_t+1) \prod_{i=1}^{t} m_i!\,,$$

where Δ is 1 or 2 depending on the symmetry, and $\lambda_i = \Sigma_t' 1$ $(i = 2, 3, \ldots, \sigma)$, where the prime indicates that the summation over the bristles includes only those with multiplicity greater than or equal to $(i-1)$. The resulting expression may be rewritten more conveniently as

$$b(n; m_1, m_2, \ldots, m_t) = \frac{q}{\Delta(m_1+1)(m_t+1)} \binom{n-1}{b-1} (q-1)^{n-b} \prod_{i=1}^{t} (m_i+1) \binom{q-1}{m_i+1}. \tag{4.26}$$

For the hypercubical lattices, it follows from the general Bethe result (4.24) that for sufficiently large n

$$g_n = n!\,2^n \binom{n-1}{b-1} R(G) \binom{d}{n} + (n-1)!\,2^{n-1}(n^2-2n-\lambda) \binom{n-1}{b-1} R(G) \binom{d}{n-1} + \ldots$$
$$\times \left(\lambda = -1 + \sum_{i=2}^{\sigma} (i-1)\lambda_i \right) \tag{4.27}$$

for weak embeddings and

$$G_n = n!\,2^n \binom{n-1}{b-1} R(G) \binom{d}{n} + \ldots \tag{4.28}$$

for strong embeddings. (Note that on making the substitutions appropriate for stars in (4.27) and (4.28) one regains the results (4.19) and (4.22), respectively.) As expected the results (4.27) and (4.28) give $1/\sigma$ expansions for the appropriate growth factors which agree with the $1/\sigma$ expansions for μ and υ through first and zeroth orders, respectively.

To conclude this section we use the data given in (4.7) and (4.12) to study the asymptotic forms of $S(n; 3)$ and $s(n; 3)$ in $d=4$ dimensions. The aim here is to test the validity of our conjecture (1.19) at the critical dimension where, according to (1.14) and (1.21), we expect to find $\gamma_{s,3}=\gamma_{S,3}=3$ (or b more generally). We have employed the same series analysis techniques as described in § 3. Biased estimates of $\gamma_{s,3}$ and $\gamma_{S,3}$ have been calculated using $\mu(d=4)=6.768\pm0.002$ (Fisher and Gaunt 1964) and $\upsilon(d=4)=5.93\pm0.03$ and are plotted against $1/n$ in figure 9. (We obtained the estimate of υ from a ratio analysis of the data given in (4.1).) Clearly these results are consistent with the conjectured values of $\gamma_{s,3}=\gamma_{S,3}=3$.

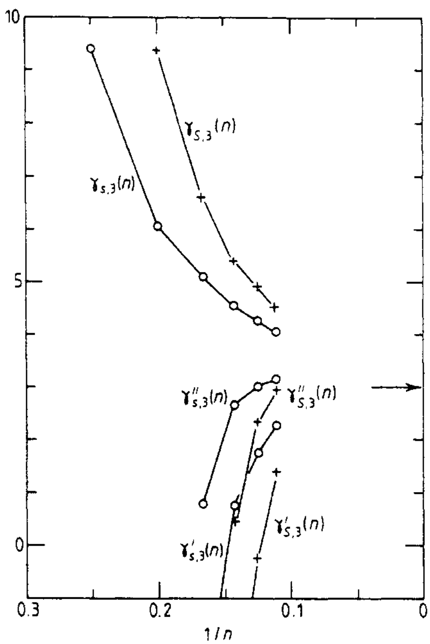

Figure 9. Plots against $1/n$ of biased ratio estimates of $\gamma_{s,3}$ and $\gamma_{S,3}$ for weak (O) and strong (+) embeddings, respectively, of stars with $b=3$ branches on the simple hypercubic lattice of $d=4$ dimensions. The conjectured exponent b is indicated by an arrow.

5. Connection with lattice trees

In this section we focus on the relationship between lattice trees with specified topologies and the total number of unrestricted lattice trees. We work with weakly embeddable clusters, although analogous ideas hold for strong embeddings.

It seems that trees with any specified topology all have the same growth constant μ, namely, the SAW limit. For unrestricted trees, on the other hand, the growth constant is λ_0, where it is known rigorously that $\lambda_0>\mu$ (Gaunt *et al* 1982). In addition, and more importantly, the universality class of trees with a specified topology depends on the number (b) of branches, possibly through the critical exponent $(\gamma+b-2)$, while for unrestricted trees the analogous exponent is $-\theta_0$. By contrast, position space renormalisation group calculations (Family 1980) suggest that varying the branching fugacity should not change the universality class of branched polymers with no cycles.

In order to reconcile these apparently contradictory results, we present the following heuristic argument. This parallels rather closely that used by Whittington *et al* (1983) to reconcile their exact enumeration results for animals with a prescribed number of cycles with the field-theoretic renormalisation group prediction that critical exponents are independent of cycle fugacity.

Let us begin by defining an activity (or fugacity) x_i associated with a vertex of degree i $(i=1,2,\ldots,q)$. Clusters with n sites and vertex set $(n_3, n_4, \ldots, n_q)$ will then be associated with an embedding factor $t(n; n_3, n_4, \ldots, n_q)$ and an activity factor $\Pi_{i=1}^{q} x_i^{n_i}$, where n_1 and n_2 are given by (1.9) and (1.17), respectively. Summing over all possible vertex sets $\{n_i\}$ and substituting for n_1 and n_2 gives the generating function

$$G(n; x_1, x_2, \ldots, x_q)$$

$$= \sum_{\{n_i\}} t(n; n_3, n_4, \ldots, n_q) \prod_{i=1}^{q} x_i^{n_i} \tag{5.1}$$

$$= x_1 x_2^{n-1} \sum_{\{n_i\}} t(n; n_3 n_4, \ldots, n_q) \frac{x_1^{1+\Sigma_{i=3}^{q}(i-2)n_i} \Pi_{i=3}^{q} x_i^{n_i}}{x_2^{1+\Sigma_{i=3}^{q}(i-1)n_i}}. \tag{5.2}$$

If we define the reduced activities

$$z_i = x_i / x_2 \qquad (i = 1, 3, 4, \ldots, q), \tag{5.3}$$

then we obtain

$$G(n; z_1, x_2, z_3, \ldots, z_q) = z_1 x_2^n \sum_{\{n_i\}} t(n; n_3, n_4, \ldots, n_q) z_1^{1+\Sigma_{i=3}^{q}(i-2)n_i} \prod_{i=3}^{q} z_i^{n_i}. \tag{5.4}$$

Letting

$$z_1 = z_3 = z_4 = \ldots = z_q = z \tag{5.5}$$

now gives

$$G(n; z, x_2) = z x_2^n \sum_{\{n_i\}} t(n; n_3, n_4, \ldots, n_q) z^b \tag{5.6}$$

where we have used the relation (1.10). The variable z, which keeps track of the number of branches in the cluster, will be referred to as the *branching fugacity*.

If we use (1.18) and (1.19) to replace $t(n; n_3, n_4, \ldots, n_q)$ in (5.6) by its dominant asymptotic form, we obtain

$$G(n; z, x_2) = z x_2^n n^{\gamma-2} \mu^n \sum_{b=1}^{b_{\max}} A_b n^b z^b \tag{5.7}$$

$$\sim z x_2^n n^{\gamma-2} \mu^n A(nz), \qquad n \to \infty, \tag{5.8}$$

where

$$A(w) = \sum_{b=1}^{\infty} A_b w^b \tag{5.9}$$

is the generating function of the amplitudes A_b. (Note that the amplitude A_b is the sum of the amplitudes associated with the cluster numbers for all the vertex sets $(n_3, n_4, \ldots, n_q)$ consistent with a given number, b, of branches.)

We see from (5.1) that $G(n; 1, 1) = a_{n0}$, the number of weakly embeddable lattice trees with n sites. If we assume (Gaunt *et al* 1982) that

$$a_{n0} \sim t_0 n^{-\theta_0} \lambda_0^n, \qquad n \to \infty, \tag{5.10}$$

then it follows that

$$A(w) \sim t_0 w^{2-\gamma-\theta_0} \mathrm{e}^{\alpha w}, \qquad w \to \infty, \tag{5.11}$$

where

$$\alpha = \ln(\lambda_0/\mu). \tag{5.12}$$

On substituting this form for $A(w)$ into (5.8) we obtain, for z arbitrary but close to the lattice tree limit $(z = 1)$,

$$G(n; z, x_2) \sim (t_0 z^{3-\gamma-\theta_0}) n^{-\theta_0} (x_2 \mu\, \mathrm{e}^{\alpha z})^n. \tag{5.13}$$

As expected, the growth constant and the critical amplitude are both functions of z. However, the critical exponent is independent of z and equal to the tree exponent θ_0. Thus, the universality class for branched polymers without cycles is independent of the branching fugacity (at least in the vicinity of the tree limit).

Within the context of this heuristic theory, the central role played by the conjecture (1.19) in deriving the above results should be noted. For example, suppose more generally that $\gamma_t = \gamma + b - 1 + g(b)$, where $g(b)$ is an arbitrary function of b. Then (5.7) becomes

$$G(n; z, x_2) = z x_2^n n^{\gamma-2} \mu^n \sum_{b=1}^{b_{\max}} A_b n^{g(b)} (nz)^b. \tag{5.14}$$

If the above kind of argument is to work at all, then we must have $g(b) = \text{constant} \equiv g$, say. Since γ_t must equal γ for $b = 1$, we are automatically led to $g(1) = g = 0$ and hence to the conjecture (1.19) for γ_t.

6. Discussion and summary

In this paper we have considered the problem of lattice trees with a specified topology. For the simple examples studied here, the number (n) of vertices in the cluster and the numbers (n_i) of branching points of degree i $(i = 3, 4, \ldots, q)$ specify the topology uniquely. Lipson and Whittington (1983) have proved rigorously that weakly embeddable trees have a growth constant of μ, the SAW limit. In § 2 we have proved a similar result for strongly embeddable trees. The growth constant is then υ, the neighbour-avoiding walk limit.

In § 1 we have presented an appealingly simple (but certainly non-rigorous) argument which, nevertheless, predicts the correct growth constant. For the critical exponents, it gives $\gamma_t = \gamma_T = \gamma + b - 1$, which we have adopted as a *conjecture*. Here γ is the critical exponent for SAWs and b, the number of branches in the topology under consideration, is determined by $(n_3, n_4, \ldots, n_q)$. This conjecture is of course correct for simple chains $(b = 1)$ and satisfies the exact upper bounds derived in § 2.

In § 3 we have derived and analysed exact enumeration data and our estimates of the critical exponents are consistent with the above conjecture.

The conjectured relation reduces correctly to the Bethe result $\gamma_t = \gamma_T = b$, as may readily be verified from (4.18) and (4.24). At the critical dimension ($d_c = 4$), the Bethe result for the critical exponents is expected to be exact and in § 4 we have tested this explicitly in the case of stars with $b = 3$ branches. Also in § 4 we have derived expansions, in inverse powers of $\sigma = 2d - 1$, for the growth constants associated with various topologies and these expansions are consistent with the rigorous results proved in § 2 and by Lipson and Whittington (1983).

In § 5 we have considered the relationship between our results for lattice trees with a specified topology and unrestricted lattice trees. We have shown that the particular expression which we conjecture for γ_t is a crucial ingredient in an heuristic theory which shows how the universality class for branched polymers without cycles is determined solely by the tree exponent θ_0 and is independent of the branching fugacity (at least in the vicinity of the tree limit). This conclusion is consistent with position space renormalisation group calculations.

In conclusion, we have argued that the critical exponent associated with a lattice tree with a specified topology depends simply on the number, b, of branches and not on any other details of the topology.

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Appendix 1. Data for weak embeddings

Table A1.1. Square lattice.

n	$s(n;3)$	$s(n;4)$	$c(n;2)$	$c(n;3)$	$b(n;1,2)$	$b(n;2,2)$
3	4					
4	36	1				
5	192	12	18			
6	872	78	222		12	
7	3 508	404	1 742	76	168	2
8	13 252	1 833	10 614	1 372	1 444	30
9	47 320	7 624	55 894	14 040	9 680	272
10	163 312	29 756	264 638	108 960	55 396	1 952
11	545 580	110 768	1 163 132	706 020	283 576	11 990
12	1 784 044	397 185	4 815 952	4 041 068	1 336 456	65 778
13	5 711 504	1382 036	19 054 456	21 047 436	5 905 584	331 146
14	18 017 008	4691 614	72 551 748	101 966 068	24 789 504	1557 178
15	55 997 476					
16	172 169 884					

Table A1.2. Triangular lattice.

n	$s(n;3)$	$s(n;4)$	$s(n;5)$	$s(n;6)$	$c(n;2)$	$c(n;3)$	$b(n;1,2)$	$b(n;2,2)$
3	20							
4	252	15						
5	2 124	228	6		207			
6	14 944	2 226	102	1	4 206		330	
7	94 584	17 688	1 098	18	51 726	2 484	7 638	111
8	558 048	124 176	9 510	207	496 758	70 020	105 042	2 940
9	3 131 904	801 768	72 018	1 908	4 097 931	1 134 468	1 111 746	45 477
10	16 929 408	4 872 648	497 442	15 291	30 461 412	13 836 120	9 991 260	533 106
11	88 877 628	28 282 224	3 212 010	111 234	209 806 563		80 163 660	
12	455 812 616	158 334 465	19 694 706	753 081				
13			115 912 092	4 822 704				
14			659 861 604	29 543 226				

Table A1.3. Simple cubic lattice.

n	$s(n;3)$	$s(n;4)$	$s(n;5)$	$s(n;6)$	$c(n;2)$	$c(n;3)$	$b(n;1,2)$	$b(n;2,2)$
3	20							
4	300	15						
5	2 856	300	6		300			
6	22 792	3 534	150	1	6 924		600	
7	161 940	33 228	2 106	30	97 884	5 232	16 272	300
8	1 075 452	271 515	22 722	489	1 069 260	171 408	262 728	9 204
9	6 774 144	2 027 376	208 800	5 944	10 032 486	3 179 760	3 229 440	165 720
10	41 164 608	14 171 640	1 723 896	60 444	84 568 002	44 364 576	33 567 492	2251 224
11	242 678 340	94 350 048	13 172 592	544 728	660 420 144		310 289 376	
12	1399 051 652	604 392 375	94 937 664	4 498 264				
13			653 538 702	34 757 880				
14				254 895 345				

Appendix 2. Data for strong embeddings

Table A2.1. Square lattice.

n	$S(n;3)$	$S(n;4)$	$C(n;2)$	$C(n;3)$	$B(n;1,2)$	$B(n;2,2)$
4.	4					
5	20	1				
6	84	4	4			
7	308	18	46			
8	1 048	68	306	4	12	
9	3 312	243	1 614	84	96	2
10	10 108	800	7 166	856	576	14
11	29 756	2 552	29 018	5 968	2 752	84
12	85 756	7 824	108 714	34 408	11 888	396
13	241 416	23 437	386 890	171 848	46 904	1 716
14	670 240	68 472	1315 314	780 964	174 868	6 792
15	1830 532	196 842	4£326 290	3286 364	619 800	25 504
16	4949 880	555 932	13 802 094	13 057 992	2118 384	91 216

Table A2.2. Triangular lattice.

n	$S(n;3)$	$C(n;2)$	$C(n;3)$
4	2		
5	18		
6	108	3	
7	516	45	
8	2 232	369	6
9	8 940	2 391	102
10	34 164	13 305	1 068
11	125 580	67 104	8 640
12	448 794	314 076	59 340
13	1566 452	1389 582	361 728

Table A2.3. Simple cubic lattice.

n	$S(n;3)$	$S(n;4)$	$S(n;5)$	$S(n;6)$	$C(n;2)$	$C(n;3)$	$B(n;1,2)$	$B(n;2,2)$
4	20							
5	204	15						
6	1 572	156	6		126			
7	10 188	1 290	54	1	2 460		132	
8	60 840	8 964	438	6	28 176	1 080	3 108	18
9	339 336	57 321	3 030	45	256 824	29 184	39 528	768
10	1817 396	341 088	19 602	296	2010 744	456 756	393 816	11 694
11	9381 300	1940 448	118 530	1 854	14 290 968	5344 740	3324 528	132 444

Appendix 3. $s(n;3)$ stars weakly embeddable in the diamond (D), body-centred cubic (BCC) and face-centred cubic (FCC) lattices (McKenzie 1967)

n	D	BCC	FCC
3	4	56	220
4	36	1 176	6 780
5	216	15 600	138 432
6	1 080	173 264	2 346 856
7	4 740	1 715 688	35 727 756
8	19 404	15 877 080	506 854 812
9	75 336	139 405 088	6 839 985 144
10	282 096	1 180 737 072	88 924 734 720
11	1023 852	9 703 453 656	
12		77 978 223 624	

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